Helping Students Crack Annuity, Perpetuity, Bond, and Stock Valuation Formulas

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Abstract

Mainstream finance textbooks present valuation formulas of annuities, perpetuities, stocks, and bonds, but the texts seldom explain the story behind them, leaving students in the dark about why these formulas work. The aim of this paper is to illuminate the black box of these formulas, thus helping finance instructors and students truly understand them. Starting from the basic valuation principle, we can reach each of these seemingly daunting formulas via a few simple algebraic steps. When students reach their "Ah-ha" moment at the end of each derivation, it motivates them and subsequently boosts their interest and confidence in learning Finance.

Introduction

Business students think finance is the most challenging business subject, and the worst part about finance is those seemingly daunting valuation formulas. One reason behind this is that most current widely-used introductory finance textbooks[3] just present these formulas directly without explaining the origin of them. The rationale and development of each formula is left as a black box. Students can work out related problems by applying the formulas mechanically. However, they do not truly understand the origin of each formula, how each formula is reached, or why the formulas work.

Students may choose to use a financial calculator or Excel built-in functions to solve time value of money (TVM) and valuation problems. But how a financial calculator or Excel solves these problems still remains a mystery to the students. They may not realize that financial calculators and Excel actually use these formulas as well. The formulas are integrated into the programs running on the calculator and Excel, so the users do not see the formulas be applied directly.

The underlying principle behind all of these formulas is that the value of any financial asset is the present value of all expected cash flows from the asset. Without knowing the ins and outs of these basic valuation formulas, students view them in isolation and have trouble grasping them. Consequently, students feel frustrated and discouraged in learning Finance. Unpacking the black box of these formulas will promote students' deep understanding of the subjects and other finance concepts that are built upon them.

Time Value of Money (TVM) Building Blocks

The time value of money is one of the most important and fundamental concepts in finance theory. Understanding the time value of money is essential in making loan, investments, capital budgeting, retirement planning, and many other financial decisions. In this section, we demonstrate how the valuation formulas of annuities and perpetuities can be derived from the basic valuation principle - the value of any financial asset

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[3] See Graham and Smart (2012), Booth, Cleary, and Drake (2012), Ross, Westerfield, and Jaffe (2012), Ehrhardt and Brigham(2013), Brealey, Myers, and Allen (2013), Ross, Westerfield, and Jordan (2013), Gitman and Zutter (2014), Parrino, Kidwell, and Bates (2014), Brigham and Houston (2015), Ross, Westerfield, and Jordan (2015).

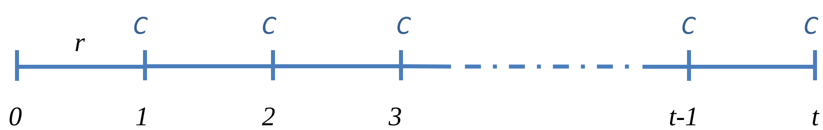
is the present value of all future cash flows expected from the asset. The annuity and perpetuity formulas are building blocks for the valuation of other financial securities. We show how stock and bond valuation formulas can be derived from annuity and perpetuity formulas in the "Putting the TVM Building Blocks to Work" section.

The derivations below follow the notations used in Ross, Westerfield, and Jordan (2015), which is a widely used introductory finance textbook in the United States and abroad.

Annuity Valuation

An annuity is a series of cash flows or payments that occur at equal intervals over a period of time. In most cases, annuity payments are equal and can be made at either the beginning or the ending of each period. If the cash flows grow at a constant rate, we would have a growing annuity. We start with the derivation of the present value of an ordinary annuity, in which the payments are made at the end of each period, as shown in Figure 1. Growing annuity is discussed in the next section.

Figure 1. Timeline of Cash Flows from an Ordinary Annuity



Consider an ordinary annuity with a constant payment of C for t periods and an interest rate of r. The value of an ordinary annuity is simply the present value (PV) of all annuity payments. Mathematically,

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^{t-1}} + \frac{C}{(1+r)^t} \tag{1}$$

We factor out $\frac{1}{1+r}$ starting from the second term on the right-hand side:

$$PV = \frac{C}{1+r} + \frac{1}{1+r}\left[\frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^{t-1}}\right] \tag{2}$$

Notice that the expression in the brackets is just the right-hand side of Eq. (1) without the last term, which is: $PV - \frac{C}{(1+r)^t}$. So Eq. (2) can be written as

$$PV = \frac{C}{1+r} + \frac{1}{1+r}\left[PV - \frac{C}{(1+r)^t}\right] \tag{3}$$

Multiplying both sides by $1 + r$ gives us:

$$(1+r)PV = C + PV - \frac{C}{(1+r)^t} \tag{4}$$

Expand the left-hand side:

$$PV + rPV = C + PV - \frac{C}{(1+r)^t} \tag{5}$$

Subtract PV from both sides and factor out C on the right-hand side, we now have:

$$rPV = C - \frac{C}{(1+r)^t} = C\,\{1 - [\frac{1}{(1+r)^t}\} \tag{6}$$

Finally, we divide both sides by r, and the final formula for the present value of an ordinary annuity is reached.

$$PV = C\,\frac{1-[1/(1+r)^t]}{r} \tag{7}$$

$\frac{1-[1/(1+r)^t]}{r}$ is often referred to as the present value interest factor for annuities (PVIF) in Finance textbooks.

Similarly, we can derive the future value of an ordinary annuity. First, we sum up the future values of all cash flows at time t. That is,

$$FV = C(1+r)^{t-1} + C(1+r)^{t-2} + \cdots + C(1+r)^2 + C(1+r) + C \quad (8)$$

We subtract C from both sides and factor $(1+r)$ from the right-hand,

$$FV - C = (1+r)[C(1+r)^{t-2} + \cdots + C(1+r)^2 + C(1+r) + C] \quad (9)$$

Note that the expression in the bracket is exactly the same as the right-hand side of Eq. (8) without the first term, so Eq. (9) can be written as

$$FV - C = (1+r)[FV - C(1+r)^{t-1}] \quad (10)$$

Expand the right-hand side,

$$FV - C = FV + rFV - C(1+r)^t \quad (11)$$

The term FV on both sides cancel each other. After we move terms around, Eq. (11) becomes

$$rFV = C(1+r)^t - C = C[(1+r)^t - 1] \quad (12)$$

Divide both sides by r, we get the future value of an ordinary annuity.

$$FV = C\frac{(1+r)^t - 1}{r} \quad (13)$$

Alternatively, we can find the future value (FV) of an ordinary annuity using the basic TVM relationship

$$FV = PV \times (1+r)^t \quad (14)$$

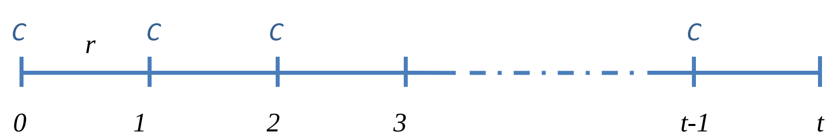
Note that we have reduced a series of annuity cash flows to a single sum equivalent at time 0 in Eq. (7). We can restate this single sum equivalent at time t to get the future value. Substitute PV in Eq. (14) with the right-hand side of Eq. (7), we obtain

$$FV = C\frac{1-\left[\frac{1}{(1+r)^t}\right]}{r} \times (1+r)^t = C\frac{(1+r)^t - 1}{r} \quad (15)$$

which is the same as Eq. (13).

For an annuity due, the payments are made at the beginning of each period. Figure 2 shows the timeline of cash flows from an annuity due.

Figure 2. Timeline of Cash Flows from an Annuity Due



The present value of an annuity due can be written as

$$PV_{Due} = C + \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^{t-1}} \quad (16)$$

We can follow similar steps to derive the present value and future value formulas for an annuity due. For brevity, we do not derive them here. We can also derive the present value of an annuity due from the present value of its corresponding ordinary annuity. Realize that each payment in an annuity due is paid one period earlier than that in an ordinary annuity. Therefore,

$$PV_{Due} = PV \times (1+r) \quad (17)$$

Replace PV with (7) in Eq. (17), we obtain:

$$PV_{Due} = C \times \frac{(1+r)-[1/(1+r)^{t-1}]}{r} \tag{18}$$

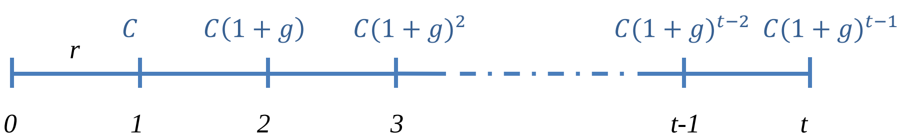
Correspondingly, the future value of an annuity due is:

$$FV_{Due} = C \times (1+r) \times \frac{(1+r)^{t}-1}{r} \tag{19}$$

Growing Annuity Valuation

A growing annuity is an annuity in which the cash flows grow at a constant rate for a specified period of time. Let C be the cash flow in the next period and g be the expected growth rate of the cash flows. The timeline for a growing annuity appears as follows:

Figure 3. Timeline of Cash Flows from a Growing Annuity



The present value of a growing annuity is then:

$$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots + \frac{C(1+g)^{t-2}}{(1+r)^{t-1}} + \frac{C(1+g)^{t-1}}{(1+r)^t} \tag{20}$$

We can factor out $\frac{1+g}{1+r}$ starting from the second term on the right-hand side:

$$PV = \frac{C}{1+r} + \frac{1+g}{1+r}\left[\frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots + \frac{C(1+g)^{t-2}}{(1+r)^{t-1}}\right] \tag{21}$$

Note that the expression in the brackets is just the PV minus the last term in Eq. (20), so Eq. (21) can be written as:

$$PV = \frac{C}{1+r} + \frac{1+g}{1+r}\left[PV - \frac{C(1+g)^{t-1}}{(1+r)^t}\right] \tag{22}$$

Multiplying both sides by $1+r$, we get:

$$(1+r)PV = C + (1+g)PV - (1+g)\frac{C(1+g)^{t-1}}{(1+r)^t} \tag{23}$$

That is:

$$PV + rPV = C + PV + gPV - \frac{C(1+g)^t}{(1+r)^t} \tag{24}$$

It is clear that PV can be cancelled on both sides. Subtract gPV from both sides and factor PV out on the left-hand side and C on the right-hand side, we get:

$$rPV - gPV = (r-g)PV = C - \frac{C(1+g)^t}{(1+r)^t} = C \times \left[1 - \left(\frac{1+g}{1+r}\right)^t\right] \tag{25}$$

We then divide both sides by $r-g$ and reach the final formula for the present value of a growing annuity:

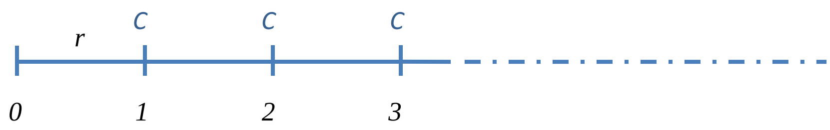
$$PV = C \times \frac{1-[(1+g)/(1+r)]^t}{r-g} \tag{26}$$

Note that this formula works for a negative g as well, in which case the annuity cash flows are decreasing at a constant rate over time (contraction).

Perpetuity Valuation

A perpetuity is a special case of annuity. In a perpetuity, the stream of equal cash payments is expected to continue forever (perpetually). Figure 4 shows the timeline for a typical perpetuity.

Figure 4. Timeline of Cash Flows from a Perpetuity



The present value of a perpetuity is then:

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots \tag{27}$$

Again, we factor $\frac{1}{1+r}$ out starting from the second term on the right-hand side of Eq. (27):

$$PV = \frac{C}{1+r} + \frac{1}{1+r}\left[\frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots\right] \tag{28}$$

Notice that the term in the brackets is exactly the same as the present value of the perpetuity in Eq. (27). Replacing it with PV, and Eq. (28) becomes

$$PV = \frac{C}{1+r} + \frac{1}{1+r}PV \tag{29}$$

Multiply both sides by $1 + r$:

$$(1 + r)PV = C + PV \tag{30}$$

Expand the left-hand side:

$$PV + rPV = C + PV \tag{31}$$

Subtract PV from both sides:

$$rPV = C \tag{32}$$

Dividing both sides by r gives us the present value formula for a perpetuity:

$$PV = \frac{C}{r} \tag{33}$$

We can also derive the perpetuity formula from the present value formula of an annuity in Eq. (7). Note that a perpetuity is a special annuity with an infinite number of payments. That is, $t = \infty$. For any reasonable discount rate r $(r > 0)$, $(1 + r)^t$ goes to infinite when t goes to infinite, and $1/(1 + r)^t$ goes to zero. Therefore,

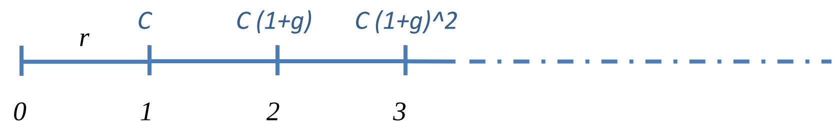
$$PV = \lim_{t\to\infty}\left(C \times \frac{1-[1/(1+r)^t]}{r}\right) = \frac{C}{r} \tag{34}$$

This is the same formula as shown in Eq. (33).

Growing Perpetuity Valuation

A growing perpetuity is a series of consecutive payments that continue indefinitely, and each payment grows at a constant rate. The timeline of a growing perpetuity with the first payment being C and the cash flow growth rate being g is shown in Figure 5.

Figure 5. Timeline of Cash Flows from a Growing Perpetuity



The present value of a growing perpetuity is:

$$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots \quad (35)$$

We can factor out the common factor $\frac{1+g}{1+r}$ starting from the second term on the right-hand side. Therefore,

$$PV = \frac{C}{1+r} + \frac{1+g}{1+r}\left[\frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots\right] \quad (36)$$

Note that the expression in the brackets is exactly the same as the right-hand side of Eq. (35), which is equal to PV. Replace the term with PV, we have:

$$PV = \frac{C}{1+r} + \frac{1+g}{1+r}PV \quad (36)$$

We then solve for PV in a similar way as equations (23) through (26).

$$(1+r)PV = C + (1+g)PV \quad (37)$$

$$PV + rPV = C + PV + gPV \quad (38)$$

$$rPV = C + gPV \quad (39)$$

$$(r-g)PV = C \quad (40)$$

Finally, we divide both sides by $r-g$ and get the present value of a growing perpetuity:

$$PV = \frac{C}{r-g} \quad (41)$$

This formula can also be derived by taking the limit of the present value of a growth annuity in Eq. (26) as t goes to infinite.

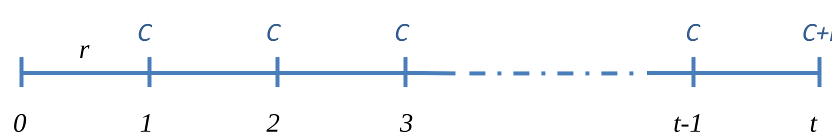
Putting the TVM Building Blocks to Work: Bond and Stock Valuations

We have derived the annuity and perpetuity formulas in the previous section. In this section, we will see how the time value of money building blocks can be put together to obtain the bond and stock valuation formulas.

Bond Valuation

A bond is a debt instrument issued by a government or corporation to raise capital. The issuer promises to pay interest periodically and principal at maturity to the bondholders according to the terms and conditions of the bond. Let us assume a bond that pays a coupon payment of *C* dollars each period, has a face value of *F* dollars and *t* periods remaining until maturity. The cash flows from this bond are illustrated in Figure 6.

Figure 6. Timeline of Cash Flows from a Typical Bond



It is clear that the cash flows from a bond can be viewed as an ordinary annuity (with equal coupon payments C) and a lump sum (the par or face value F) to be paid at maturity. So the value of a bond should be the sum of the present value of an annuity and the present value of a lump sum.

$$\text{Bond value} = \boxed{\frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^{t-1}} + \frac{C}{(1+r)^t}} + \boxed{\frac{F}{(1+r)^t}}$$

PV of an Ordinary Annuity + PV of a Lump Sum

We have derived the present value formula of an ordinary annuity which is shown in Eq. (7). For a payment of F dollars that will be made t periods from now, its present value is simply $\frac{F}{(1+r)^t}$. Therefore,

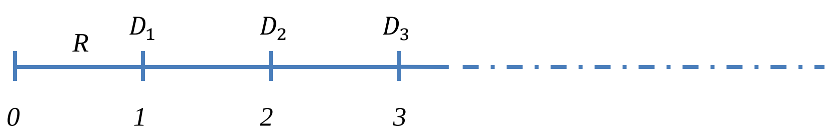
$$\text{Bond value} = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^{t-1}} + \frac{C}{(1+r)^t} + \frac{F}{(1+r)^t}$$

$$= C \times \frac{1-1/(1+r)^t}{r} + \frac{F}{(1+r)^t} \tag{42}$$

Stock Valuation

A stock is a share of ownership in a corporation. Companies issue stocks as a means of raising money to fund further business growth. The issuing company is expected to pay dividends to its shareholders. Assume the dividend payments from a hypothetical stock are as follows:

Figure 7. Timeline of Cash Flows from a Hypothetical Stock



The price of a stock is the present value of all its future dividend payments. For a stock that pays dividends of D_1 in period 1, D_2 in period 2, D_3 in period 3, etc., the price of the stock is then:

$$P_0 = \frac{D_1}{1+R} + \frac{D_2}{(1+R)^2} + \frac{D_3}{(1+R)^3} + \cdots \tag{43}$$

where R is the required rate of return on the stock.

We now make some assumptions about the pattern of future dividend payments. Following the mainstream introductory corporate finance textbooks, we consider the following three special cases.

Zero Growth

If a stock pays a constant dividend, that is, the dividend growth rate is zero, then

$$D_1 = D_2 = D_3 = \cdots = D \tag{44}$$

In this case, the dividend stream can be viewed as an ordinary perpetuity. Using the perpetuity valuation formula in Eq. (33), we conclude that the price of the stock would be:

$$P_0 = \frac{D}{R} \tag{45}$$

This is the formula to calculate the price of a preferred stock, as preferred stocks pay constant dividends over time.

Constant Growth (Dividend Growth Model)

In the Dividend Growth Model, we assume the dividends grow at a constant rate g. Therefore,

$$D_2 = D_1(1+g) \tag{46}$$

$$D_3 = D_2(1+g) = D_1(1+g)^2 \tag{47}$$

$$\dots \dots$$

$$D_t = D_{t-1}(1+g) = D_1(1+g)^{t-1} \tag{48}$$

Substituting these dividends in Eq. (43), we have:

$$P_0 = \frac{D_1}{1+r} + \frac{D_1(1+g)}{(1+r)^2} + \frac{D_1(1+g)^2}{(1+r)^3} + \cdots \tag{49}$$

Comparing Eq. (47) with Eq. (35), we should realize that a stock paying dividends that grow at a constant rate is actually a growing perpetuity. So we apply the present value of a growing perpetuity formula (41) here and get the Dividend Growth Model:

$$P_0 = \frac{D_1}{R-g} \tag{50}$$

If we know the dividend just paid, D_0, instead, the expected dividend payment for the next period is then $D_1 = D_0(1+g)$. So the stock price can be written as:

$$P_0 = \frac{D_0(1+g)}{R-g} \tag{51}$$

We can, in turn, find the expected return of a stock from its observed market price P_0 using the dividend growth model. Multiplying both sides of Eq. (50) by $\frac{R-g}{P_0}$ gives us:

$$R - g = \frac{D_1}{P_0} \tag{52}$$

Add g to both sides to get the expected return:

$$R = \frac{D_1}{P_0} + g \tag{53}$$

Alternatively, if we know D_0 instead of D_1, the expected return is then:

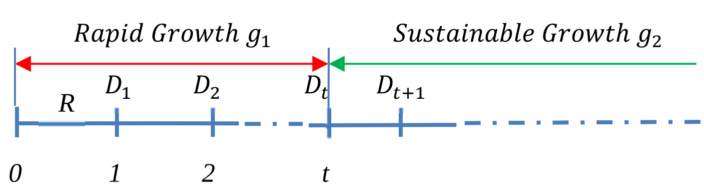
$$R = \frac{D_0(1+g)}{P_0} + g \tag{54}$$

The first term on the right-hand side of Eqs. (53) and (54) is called the dividend yield. The second part of the expected return is the dividend growth rate, g. It is also the rate at which the stock price grows. Thus, the dividend growth rate can also be interpreted as the capital gains yield.

Non-constant Growth (Supernormal Growth Model)

Some firms may experience higher than normal growth in dividend payments for a limited time. After this supernormal growth period, the dividend is expected to grow at a sustainable rate. Consider the case of a firm whose dividend grows at a faster rate g_1 for t periods and then grows at a more sustainable rate g_2 thereafter, as illustrated in Figure 8.

Figure 8. Non-constant Dividend Growth



Clearly, the dividend stream after t years is a growing perpetuity. Applying the growth perpetuity formula (41) or dividend growth model (50) or (51), we find the value of all future dividend payments starting from D_{t+1} at period t:

$$P_t = \frac{D_{t+1}}{R-g_2} = \frac{D_t(1+g_2)}{R-g_2} \tag{55}$$

This is called the terminal value or horizon value. We can now calculate the value of the stock at time 0 as the present value of dividend payments in the first t periods plus the present value of the terminal value.

$$P_0 = \frac{D_1}{1+R} + \frac{D_2}{(1+R)^2} + \cdots + \frac{D_t}{(1+R)^t} + \frac{P_t}{(1+R)^t} \tag{56}$$

Conclusion

The annuity, perpetuity, bond, and stock valuation formulas seem daunting to many business and finance students. Finance instructors often follow textbooks to present these formulas to students directly without explaining the story behind these formulas, leaving students in the dark to wonder why and how these formulas work. Mainstream finance textbooks should at least devote a little more text to explaining the origin of these formulas. The goal of this paper is to illuminate the black box, thus helping finance instructors and students to truly understand annuity, perpetuity, bond, and stock valuation formulas. The main section of this paper can be included as an appendix in introductory finance textbooks.

Instructors should remind students that no matter how complex a valuation formula seems at first glance, it is all built upon the basic time value of money building blocks. Spending a few minutes of class time deriving the valuation formulas will alleviate students' anxiety and fear of these formulas. At the end of each derivation, students can experience some great "Ah-ha" moments and subsequently become more motivated. Going through the derivations not only helps students to truly understand these formulas, but also sharpens students' quantitative skills and boosts their confidence and interest in learning finance.

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